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Bernoulli Shifts Induce Bernoulli Shifts

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1. INTRODUCTION

A Bernoulli shift is an invertible measure preserving transformation on the unit interval with Lebesgue measure that admits an independent generator, i.e., there exists a partition of the unit interval into a finite or countable number of disjoint sets of positive measure such that distinct iterates under the transformation of sets in the partition are independent and the sigma algebra generated by all the iterates of sets in the partition is the sigma algebra of all measurable sets (see Section 2). D. S. Ornstein has shown that Bernoulli shifts with the same entropy are isomorphic [9, 10, 12].

In [4] it was shown that each ergodic measure preserving transformation induces mixing transformations on a dense class of subsets (see Section 2). In general an ergodic measure preserving transformation cannot induce Bernoulli shifts since they have positive entropy [1, 2]. Our purpose is to prove that each Bernoulli shift, with finite or infinite entropy, induces Bernoulli shifts on a dense class of subsets. It is an open question whether each transformation with positive entropy induces Bernoulli shifts.

By Ornstein's isomorphism theorem, it suffices to prove the above result for one Bernoulli shift with entropy h for each h, $0 < h \leqslant \infty$. The construction in [5–7] is generalized to obtain a mixing Markov shift with the desired property. Since mixing Markov shifts are isomorphic to Bernoulli shifts [3], the result follows.

In [9] *P.* Shields showed that the transformations constructed in [7] were Bernoulli shifts by applying the result that an increasing family of Bernoulli shifts is a Bernoulli shift [10]. We shall also apply this result to the generalized construction. The main point of the construction is that it yields a Bernoulli shift with specified entropy for which it is

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relatively easy to see how the induced transformations behave on a sufficiently large class of subsets.

2. Preliminaries

Let $(X, \mathcal{A}, m)$ denote the unit interval with Lebesgue measure and let τ be an invertible point transformation mapping X onto X. τ is *measure preserving* (m.p.) if $m(\tau A) = m(A) = m(\tau^{-1}A)$, $A \in \mathcal{A}$. τ is *ergodic* if $m(A) > 0$ and $\tau A \subset A$ implies $m(A) = 1$. A metric topology is defined on $\mathcal{A}$ by the distance function $m(A \Delta B)$.

Given a subset A of positive measure, we consider the induced measure space $(A, \mathcal{A}_A, m_A)$, where $\mathcal{A}_A = \{B \subset A: B \in \mathcal{A}\}$ and $m_A(B) = m(B)/m(A)$. The induced transformation τ_A is defined on A as follows. For a.e. $x \in A$ there exists a smallest positive integer $n = n(x)$ such that $\tau^n(x) \in A$. Define $\tau_A(x) = \tau^n(x)$. This definition is due to Kakutani [8]. It follows that τ_A is ergodic and measure preserving on A since τ is ergodic and measure preserving on X.

A subclass of $\mathcal{A}$ is *independent* if the measure of the intersection of any finite number of sets in the subclass is equal to the product of their measures. $P = \{p: p \in P\}$ is a *partition* if P is a finite or countable collection of disjoint sets of positive measure whose union is X. P is a *generator for* τ if $\mathcal{A}$ is the sigma algebra generated by the class $\{\tau^i p: p \in P,\ i = 0, \pm 1, \pm 2,...\}$. P is an independent generator if each class $\{\tau^i p_i : p_i \in P,\ 0 \leqslant i \leqslant n\}$ is independent for every $n = 1, 2, 3,...$ A. Bernoulli shift τ with independent generator P has *entropy* $h(\tau) = -\sum_{p \in P} m(p) \log m(p)$ [2]. The entropy of an induced transformation is $h(\tau_A) = h(\tau)/m(A)$ (Abramov [1]).

Let $P_n = \bigvee_{i=0}^{n} \tau^i P$ denote the common refinement of the partitions $\tau^i P$, $0 \leqslant i \leqslant n$. Thus $A \in P_n$ implies $A = \bigcap_{i=0}^{n} \tau^i p_i$, where $p_i \in P$, $0 \leqslant i \leqslant n$. Let $\alpha(p, q) \geqslant 0$, $p,q \in P$; $m(q) = \sum_{p \in P} \alpha(p, q) m(p)$, $q \in P$; and $\sum_{q \in P} \alpha(p, q) = 1$, $p \in P$. We say P is a *Markov* partition for τ if for all A as above and $q \in P$,

$$m(\tau A \cap q) = \alpha(p_0, q)\, m(A).$$

τ is *mixing* if

$$\lim_n m(\tau^n A \cap B) = m(A)\, m(B), \qquad A, B \in \mathcal{A}.$$

A transformation is a *Markov shift* if the transformation admits a Markov generator P. In this case the entropy of τ is

$$h(\tau) = -\sum_{p \in P} m(p) \sum_{q \in P} \alpha(p, q) \log \alpha(p, q).$$

A Markov shift is a Bernoulli shift when $\alpha(p, q) = m(q)$, p, $q \in P$.

3. Construction

A *column* C is an ordered set of disjoint intervals $I_1, \ldots, I_h$ that are left-closed, right open, and have the same measure. C has *height* $h(C) = h$, *width* $w(C) = m(I_1)$, *base* $B(C) = I_1$, and *top* $F(C) = I_h$. Let $C^* = \bigcup_{i=1}^{h} I_i$. C_1 and C_2 are *disjoint columns* if $C_1{}^*$ and $C_2{}^*$ are disjoint sets. A column C generates a one-to-one mapping τ_C of $C^* - I_h$ onto $C^* - I_1$, where τ_C maps I_i linearly onto I_{i+1}, $1 \leqslant i < h$. Thus if C has height h and base I, we can express C as

$$C = (\tau_C{}^i(I) \colon 0 \leqslant i < h).$$

Let J be a subinterval of I. Then

$$C_J = (\tau_C{}^i(J) \colon 0 \leqslant i < h)$$

is a *subcolumn* of C.

A *tower* T is an ordered set of disjoint columns with possibly different heights and widths. $T = (C_j : 1 \leqslant j \leqslant k)$ has *base* $B(T) = \bigcup_{j=1}^{k} B(C_j)$, *top* $F(T) = \bigcup_{j=1}^{k} F(C_j)$, *width* $w(T) = m(B(T))$, and $T^* = \bigcup_{j=1}^{k} C_j{}^*$. T generates the mapping τ_T consisting of the mappings τ_C for C in T. In general, τ_T is measure preserving since τ_C is measure preserving for each column C.

Let $T = (C_j : 1 \leqslant j \leqslant k)$ and $T' = (C_j{}' \colon 1 \leqslant j \leqslant k)$. Let $0 < \alpha \leqslant 1$. T' is an *α-copy of* T if $C_j{}'$ is a subcolumn of C_j and $w(C_j{}') = \alpha w(C_j)$, $1 \leqslant j \leqslant k$. In this case we denote $T' = \alpha T$.

Given $\alpha_i > 0$, $1 \leqslant i \leqslant r$, and $\sum_{i=1}^{r} \alpha_i = 1$, we can decompose T into disjoint copies $\alpha_i T$, $1 \leqslant i \leqslant r$, by cutting each column C_j in T into subcolumns of widths $\alpha_i w(C_j)$, $1 \leqslant i \leqslant r$, and then grouping the columns of width $\alpha_i w(C_j)$, $1 \leqslant j \leqslant k$, to form $\alpha_i T$.

Given disjoint towers T_1 and T_2, let $T_1 + T_2$ be the tower consisting of the columns in T_1 followed by the columns in T_2. We have $B(T_1 + T_2) = B(T_1) \cup B(T_2)$, $F(T_1 + T_2) = F(T_1) \cup F(T_2)$, and

$(T_1 + T_2)^* = T_1^* \cup T_2^*$. We can picture $T_1 + T_2$ as T_2 placed to the right of T_1. Note that $\frac{1}{2}T + \frac{1}{2}T$ has twice as many columns as T.

Given disjoint columns C_1 and C_2 with the same width, let $C_1 * C_2$ be the column consisting of the intervals in C_1 followed by the intervals in C_2. Thus $B(C_1 * C_2) = B(C_1), F(C_1 * C_2) = F(C_2)$, and $h(C_1 * C_2) = h(C_1) + h(C_2)$.

Given a column C and $\alpha_j > 0$, $1 \leqslant j \leqslant k$, $\sum_{j=1}^{k} \alpha_j = 1$, we can form disjoint subcolumns $\alpha_j C$ of width $\alpha_j w(C)$, $1 \leqslant j \leqslant k$.

Now let $T = (C_j : 1 \leqslant j \leqslant k)$ with T^* and C^* disjoint and $w(T) = w(C)$. Let $\alpha_j = w(C_j)/w(T)$, $1 \leqslant j \leqslant k$, and define

$$C * T = (\alpha_j C * C_j \colon 1 \leqslant j \leqslant k).$$

Briefly, C^*T consists of T placed above C.

Next, let $T_1 = (C_j : 1 \leqslant j \leqslant k)$ and let T_2 be a tower such that $w(T_1) = w(T_2)$. Let $\alpha_j = w(C_j)/w(T_1)$ and let $\alpha_j T_2$ be disjoint copies of T_2, $1 \leqslant j \leqslant k$. Define

$$T_1 * T_2 = \sum_{j=1}^{k} C_j * \alpha_j T_2 .$$

Thus $T_1 * T_2$ consists of a copy of T_2 above each column in T_1.

Let T be a tower and let $(\frac{1}{2}T)_0$ and $(\frac{1}{2}T)_1$ be disjoint copies of T. To be precise, if $[a, a + 2b)$ is the base of the j-th column in T, then we let $[a, a + b)$ be the base of the j-th column in $(\frac{1}{2}T)_0$ and let $[a + b, a + 2b)$ be the base of the j-th column in $(\frac{1}{2}T)_1$. Define a tower $S(T)$ as

$$S(T) = (\tfrac{1}{2}T)_0 * (\tfrac{1}{2}T)_1 .$$

Thus $S(T)$ consists of a half-size copy $(\frac{1}{2}T)_0$ of T and a copy of T above each column in $(\frac{1}{2}T)_0$. $(\frac{1}{2}T)_0$ is referred to as the copy of rank 0. $(\frac{1}{2}T)_1$ is decomposed into k copies, assuming T has k columns, which are referred to as copies of rank 1. Thus each column of rank 0 has a copy of rank 1 above it.

Proceeding inductively, $S^u(T)$ consists of one copy of T of rank 0, k copies of T of rank 1, k^2 copies of T of rank 2,..., k^{2u-1} copies of T of rank $2u - 1$. Each column in a copy of rank r has a copy of rank $r + 1$ above it, $0 \leqslant \ < 2u + 1$. In this sense, we speak of the copies being stacked independently.

Let n be a positive integer and let $T_i = (1/n)T$, $1 \leqslant i \leqslant n$, be n disjoint copies of T. Define

$$S_n(T) = S\left(\sum_{i=1}^{n} T_i\right),$$

$$= \left(\tfrac{1}{2}\sum_{i=1}^{n} T_i\right) * \left(\tfrac{1}{2}\sum_{i=1}^{n} T_i\right).$$

In this case $\frac{1}{2}\sum_{k=1}^{n} T_i$ consists of n copies of T which we call copies of rank 0. Above each column in a copy of rank 0 is a copy of $\frac{1}{2}\sum_{i=1}^{n} T_i$ which consists of n copies of T, which we call copies of rank 1. There are n^2k copies of rank 1 when T has k columns.

Given T and a sequence n_k, $k = 1, 2,\ldots$, denote $T^1 = T$ and $T^{k+1} = S_{n_k}(T^k)$, $k \geqslant 1$. In general, T^{k+1} will consist of copies of T of rank r, $0 \leqslant r < 2k$. Each column in a copy of rank r will have copies of rank $r+1$ above it, $0 \leqslant r < 2k$. Note that $\tau_{T^{k+1}}$ extends τ_{T^k} to one-half of the top $F(T^k)$, and $\tau_{T^{k+1}}^{-1}$ extends $\tau_{T^k}^{-1}$ to one-half of the base $B(T^k)$. Also each extension is measure preserving.

It follows that $\tau = \tau(T, (n_k)) = \lim_k \tau_{T^k}$ is defined a.e. on X in a one-to-one measure preserving manner. In case $n_k = 1$ and T has columns of equal width, then $\tau = \tau(T)$ as in [5–7]. If n_k is chosen sufficiently large, then we shall show that $h(\tau) = \infty$.

4. Main Results

Let P denote the partition consisting of the intervals in the columns in T. Let $p \in P$ be an interval that is not a top interval in a column in T. If q is the interval above p, then $\alpha(p, q) = 1$, and $\alpha(p, q) = 0$ otherwise. Let p be the top interval in a column. If q is the base interval of a column, then $\alpha(p, q) = m(q)/w(T)$, and $\alpha(p, q) = 0$ otherwise. We shall prove that P is a Markov partition for τ with respect to the transition probabilities $\alpha(p, q)$, p, $q \in P$. This is a consequence of Lemma 2 below, which generalizes Lemma 3.5 [7]. In this paper we utilize [10] and need not first prove that τ is ergodic as in [7].

Let $P_n = \bigvee_0^{n-1} \tau^i P$ and let $P_n(p)$ denote the class of sets in P_n that are contained in p, $p \in P$. If $A \in P_n$, then $A = \bigcap_0^{n-1} \tau^i p_i$ where $p_i \in P$, $0 \leqslant i < n$; hence $A \in P_n(p_0)$. If $B \in P_{n+1}$, then $B = \bigcap_0^{n} \tau^i p_i = \tau(\bigcap_0^{n-1} \tau^i p_{i+1}) \cap P_0$; hence $B = \tau A \cap p_0$, where $A \in P_n$ and $p_0 \in P$.

LEMMA 1. *Let $u \geqslant 0$, $p \in P$, and $A \in P_u(p)$. Given $\epsilon > 0$, there exists $N = N(u, \epsilon)$ such that $n \geqslant N$ implies $A = E \cup G$ where E is a union of disjoint subintervals of p in copies of T in T^n and G is a subset of $\bigcup_{i=0}^{u-1} \tau^i B(T^n)$ with $m(G) < \epsilon$.*

Proof. Let $u = 0$; hence $A = p$. In this case the conclusion holds for $N = 1$ and $\epsilon = 0$ since p is distributed as subintervals in the copies of T in T^n. Assume the result holds for k and all $\epsilon > 0$. Consider $N = N(k+1, \epsilon) \geqslant N(k, \epsilon/2)$ such that $1/2^N < \epsilon/2$. If $A' \in P_{k+1}(q)$, then $A' = \tau A \cap q$, where $A \in P_k(p)$ for some p. Thus $A = E \cup G$ as above. If p is not the top interval in a column in P, then $\tau(p) = q' \in P$. Hence if I is an interval in E in a copy of T in T^n then $\tau I = J$ is the interval above I in the same copy of T and J is a subinterval of q'. In this case $A' \in P_{k+1}(q')$ and the result holds by measure preservance where $A' = E' \cup G'$ with $E' = \tau E$ and $G' = \tau G$.

Now suppose p is the top interval of a column in P. Then each interval I in E is the top interval of the corresponding column in a copy of T in T^n. Hence if I is in a copy of rank $r < 2n - 1$, then τI is the base of a copy of T of rank $r + 1$ in T^n. Hence $\tau I \cap q$ is a subinterval of q in a copy of T of rank $r + 1$, where q must be the base of a column in T. We ignore I if I is in a copy of rank $2n - 1$; hence $I \subset F(T^n)$ in this case. Therefore $\tau(I) \subset B(T^n)$ and $m(B(T^n)) < 1/2^n < \epsilon/2$. Let $E_1 = \cup I$ for $I \subset F(T^n)$. In this case the result holds by measure preservance since $A' = E' \cup G'$, where $E' = \tau(E - E_1)$ and $G' = \tau G \cup \tau E_1$. Thus we have the conclusion for $k + 1$ and $\epsilon > 0$; hence induction completes the proof.

LEMMA 2. *P is a Markov operator for τ where*

$$\alpha(p, q) = 1, \qquad p \not\subset F(T),\ q = \tau p; \tag{2.1}$$

$$\alpha(p, q) = 0, \qquad p \not\subset F(T),\ q \neq \tau p; \tag{2.2}$$

$$\alpha(p, q) = m(q)/w(T), \qquad p \subset F(T),\ q \subset B(T); \tag{2.3}$$

$$\alpha(p, q) = 0, \qquad p \subset F(T),\ q \not\subset B(T); \tag{2.4}$$

Proof. Let $\epsilon > 0$ and $A \in P_k(p)$. For (2.1), we note that Lemma 1 implies $\tau E \subset q$; hence $m(\tau A \cap q) = m(\tau E) + m(\tau G \cap q)$. Thus $m(A) = m(E) + m(G)$ implies

$$m(A) - \epsilon < m(\tau A \cap q) \leqslant m(A).$$

Since ϵ is arbitrary, we have (2.1). (2.2) follows from Lemma 1 since $\tau A \cap q = \varnothing$. For (2.3) we note that Lemma 1 implies $m(\tau E \cap q) = \alpha(p, q)m(E)$. Hence $\tau A \cap q = (\tau E \cap q) \cup (\tau G \cap q)$ implies

$$(m(A) - \epsilon)\,\alpha(p, q) < m(\tau A \cap q) < m(A)\,\alpha(p, q) + \epsilon.$$

Since ϵ is arbitrary, (2.3) follows. (2.4) follows from Lemma 1 since $\tau A \cap q = \varnothing$.

Let P^k denote the partition of intervals in columns of T^k.

LEMMA 3. *P^k is a Markov operator for τ where*

$$\alpha(p, q) = 1, \qquad p \not\subset F(T^k),\ q = \tau p. \tag{3.1}$$

$$\alpha(p, q) = 0, \qquad p \not\subset F(T^k),\ q \neq \tau p. \tag{3.2}$$

$$\alpha(p, q) = m(q)/w(T^k), \qquad p \subset F(T^k),\ q \subset B(T^k). \tag{3.3}$$

$$\alpha(p, q) = 0, \qquad p \subset F(T^k),\ q \not\subset B(T^k). \tag{3.4}$$

Proof. Apply Lemma 2 with P and T replaced by P^k and T^k, respectively, noting that

$$\tau = \tau(T^k, (n_i)_{i>k}).$$

A tower T is an *M-tower* if T has at least two columns C_1 and C_2 such that $h(C_1) = h(C_2) + 1$. Let $\mathcal{A}^k$ denote the σ-algebra generated by τ on P^k. It easily follows that T^k is an M-tower if T is an M-tower, $k > 1$. If T^k is an M-tower, then the greatest common divisor of cycles of the Markov chain corresponding to the transition probabilities $\alpha(p, q)$, p, $q \in P^k$ is one. Since $\alpha(p, q) > 0$, $p \in F(T^k)$, $q \in B(T^k)$, the Markov chain is ergodic. Hence τ is mixing on P^k so that τ is a mixing Markov shift on $\mathcal{A}^k$. Therefore τ is a Bernoulli shift on $\mathcal{A}^k$ [3]. Now $\mathcal{A}$ is the σ-algebra generated by $\mathcal{A}^k$, $k \geqslant 1$, since each $n \in X$ implies x is contained in arbitrarily small intervals in $\mathcal{A}^k$ for sufficiently large k. Thus τ is a Bernoulli shift on $\mathcal{A}$ by [10].

THEOREM 1. *Let T be an M-tower and let (n_k) be a sequence of positive integers. Then $\tau = \tau(T, (n_k))$ is a Bernoulli shift.*

Let $\tau/\mathcal{A}^k$ denote the restriction of τ to the σ-algebra $\mathcal{A}^k$; hence $\tau = \tau/\mathcal{A}$.

LEMMA 4. $h(\tau/\mathcal{A}^1) = -\sum_{q \subset B(T)} m(q) \log m(q)/w(T)$.

Proof. By Lemma 2 we have

$$h(\tau/\mathcal{O}\!\ell_1) = -\sum_{p\subset F(T)} m(p) \sum_{q\subset B(T)} m(q)/w(T) \log m(q)/w(T)$$

$$= -w(T) \sum_{q\subset B(T)} m(q)/w(T) \log m(q)/w(T)$$

$$= -\sum_{q\subset B(T)} m(q) \log m(q)/w(T).$$

In the same way we obtain

LEMMA 5. $h(\tau/\mathcal{O}\!\ell^k) = -\sum_{q\subset B(T^k)} m(q) \log m(q)/w(T^k)$.

If T has K columns of equal width w, then $w(T) = Kw$ and Lemma 4 implies $h(\tau/\mathcal{O}\!\ell^1) = w(T) \log K$. Now suppose we replace T by $\sum_{i=1}^{n} ((1/n)T)$. Note that $w(T) = w(\sum_{i=1}^{n} (1/n)T))$ but the number of columns is increased by a factor of n. Suppose $n = K^{N-1}$; hence the number of columns is K^N. This implies the entropy is increased by a factor of N. A straightforward computation shows that the entropies corresponding to T and $S(T)$ are the same; hence for any $n_k \geqslant 1$, the sequence $h(\tau/\mathcal{O}\!\ell^k)$ is nondecreasing. If c_k is the number of columns in T^k and $n_k = c_k$ then $T^{k+1} = S_{c_k}(T^k)$, $k \geqslant 1$. Hence the entropy is doubled at each stage.

LEMMA 6. *Let T be an M-tower with K columns of equal width and let n_k be the number of columns in T^k, $k \geqslant 1$. Then $h(\tau/\mathcal{O}\!\ell^k) = 2^{k-1}w(T)$ $\log K$, $k \geqslant 1$, and $h(\tau) = \infty$.*

Proof. $h(\tau) = \lim_k h(\tau/\mathcal{O}\!\ell^k)$.

For the case of unequal column widths, note that Lemma 4 implies

$$h(\tau/\mathcal{O}\!\ell^1) = w(T) \log w(T) - \sum_{q\subset B(T)} m(q) \log m(q).$$

If T is replaced by $\sum_{i=1}^{K^N} ((1/K^N)T)$, then q is replaced by K^N sets of measure $m(q)/K^N$. The entropy increases by $w(T)N \log K$. It follows that n_k can be chosen large with respect to $1/w(T^k)$ to guarantee $\lim_k$ $h(\tau/\mathcal{O}\!\ell^k) = \infty$.

In order to obtain τ with $h(\tau) = h$ for finite positive h, we proceed as follows. Let $h \leqslant (n/n + 1) \log n$. Choose T with n columns consisting of one interval each of width w and one column with two intervals of width $(1 - nw)/2$ each. Thus T is an M-tower and if $n_k = 1$ for all k, then $h(\tau) = h(\tau/\mathcal{A}^1)$. Lemma 4 implies

$$h(\tau) = nw \log(1 + nw)/(2nw) + (1 - nw)/2 \log(1 + nw)/(1 - nw).$$

If $w = 0$, then $h(\tau) = 0$ and if $w = 1/n + 2$, then

$$h(\tau) = [(n + 1)/(n + 2)] \log(n + 1).$$

Since $h(\tau)$ is a continuous function of w, there exists w between 0 and $1/n + 2$ such that $h(\tau) = h$.

We shall only sketch the proof of Lemma 6 below since it is the same as in [7]. Let F be a union of intervals from a tower T such that F contains at least one interval from each column of T. Let T_F be the tower T restricted to F. We can now apply the construction on F to define a transformation $\tau(T_F, (n_k))$. It is shown that $\tau(T, (n_k))_F = \tau_F = \tau(T_F, (n_k))$. Hence τ_F is Bernoulli if T_F is an M-tower. Now the partitions P^k are increasing and generate $\mathcal{A}$. It follows that given $A \in \mathcal{A}$ and $\epsilon > 0$, there exists k sufficiently large and a set F that is a union of intervals in P^k such that $m(A \Delta F) < \epsilon$. It is not difficult to guarantee that F contains $B(T^k)$ and $(T^k)_F$ is an M-tower. Replacing T by T^k in the preceding argument, it follows that τ_F is a Bernoulli shift. Thus we have

THEOREM 2. *Let T be a tower and let (n_k) be a sequence of positive integers. The sets on which $\tau = \tau(T, (n_k))$ induces Bernoulli shifts are dense in $\mathcal{A}$.*

An application of Ornstein's isomorphism theorem for Bernoulli shifts with finite or infinite entropy and Theorems 1 and 2 now yield Theorem 3.

THEOREM 3. *Each Bernoulli shift induces Bernoulli shifts on a dense class of measurable sets.*

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